Optimization based on Mixed Integer Nonlinear Programming methods

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What is MP?

What is a MINLP?
- Subclasses of MINLP
- Dealing with nonconvexities

Global Optimization methods

Spatial Branch-and-Bound
- Expression trees
- Convex relaxation
- Variable ranges
- Bounds tightening

References

How far can we get?

Practical Tools
- MINLP Solvers
- Modeling Languages
- Neos
- MINLP Libraries

Smart Grids

Outline

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What is Mathematical Programming?

- MP: formal language for expressing optimization problems P
  - **Parameters** p =problem input
    p also called an **instance** of P
  - **Decision variables** x: encode problem output
  - **Objective function** $\min f(p, x)$
  - **Constraints** $\forall i \leq m \quad g_i(p, x) \leq 0$
    f, g: explicit mathematical expressions involving symbols p, x
- If an instance p is given (i.e. an assignment of numbers to the symbols in p is known), write $f(x), g_i(x)$

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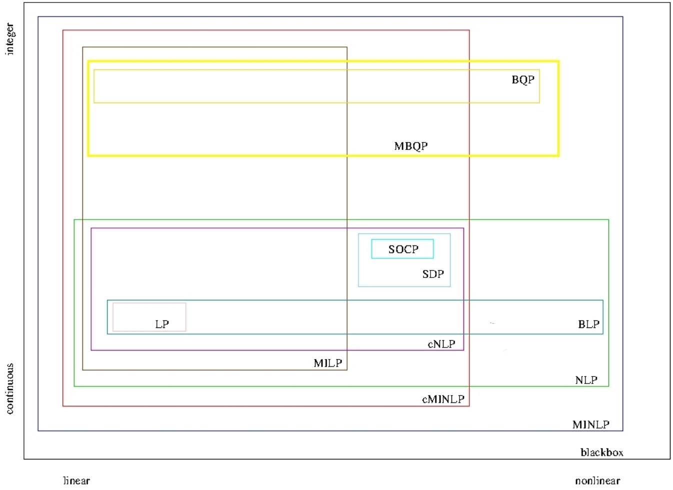
Main optimization problem classes



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What is a (nonconvex) MINLP?

Mixed Integer NonLinear Programming (MINLP).

$$\begin{array}{rcl} \min f(x,y) & & \\ g(x,y) & \leq & 0 \\ x & \in & X = \{x \mid x \in \mathbb{R}^p, Dx \leq d, x^L \leq x \leq x^U\} \\ y & \in & Y = \{y \mid y \in \mathbb{Z}^q, Ay \leq a, y^L \leq y \leq y^U\} \end{array}$$

with $f(x,y) : \mathbb{R}^{p+q} \to \mathbb{R}$ and $g(x,y) : \mathbb{R}^{p+q} \to \mathbb{R}^m$ are

* continuous

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with $f(x,y) : \mathbb{R}^{p+q} \to \mathbb{R}$ and $g(x,y) : \mathbb{R}^{p+q} \to \mathbb{R}^m$ are

* continuous
* twice differentiable

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functions.

- Local optima are not always global optima .

Interesting subclasses of MINLP

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Subclasses :

* f and g are convex: convex MINLPs.
* f and g are separable: separable MINLP.
* f and g are quadratic: quadratic MINLP.
* f and g are polynomial: polynomial MINLP.

Convex MINLP methods

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Based on:

1. Continuous (NLP) Relaxation: relax integrality requirements

Convex MINLP methods

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Based on:

1. Continuous (NLP) Relaxation: relax integrality requirements
2. (Mixed Integer) Linear Relaxation: outer approximation

Continuous (NLP) relaxation

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Continuous (NLP) relaxation

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$$

NP-hard to solve in general!

MINLP branch-and-bound

Branch-and-bound algorithm:

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NLP solver used:
Local NLP solvers $\rightarrow$ local optimum

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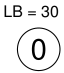

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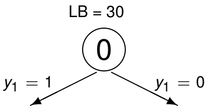

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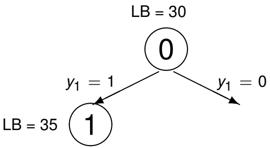

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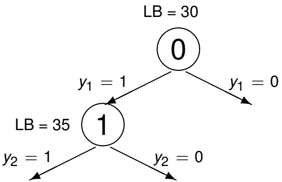

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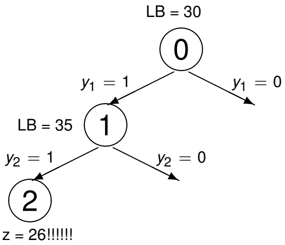

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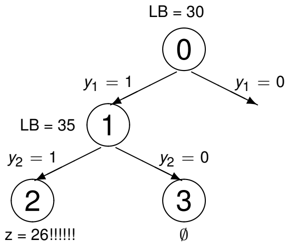

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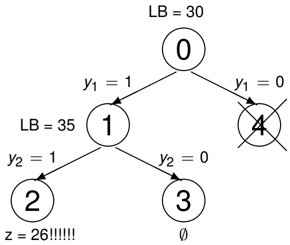

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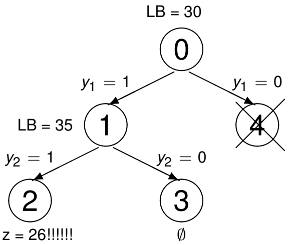

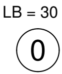

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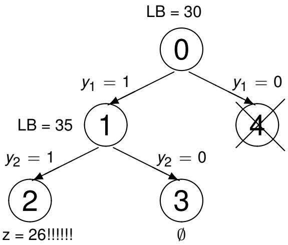

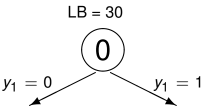

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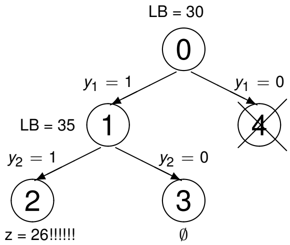

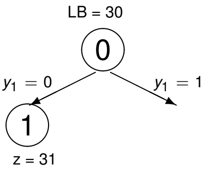

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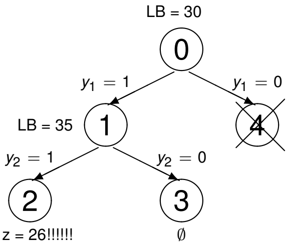

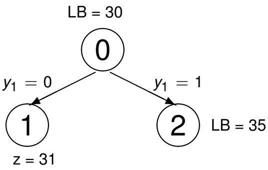

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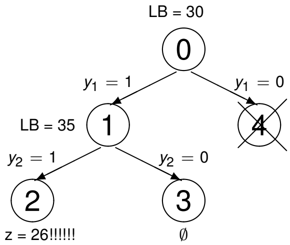

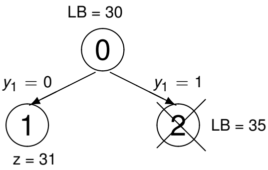

Specific options for nonconvex MINLPs

Different starting points for root/each node.

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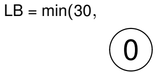

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LB = min(30, 28,

0

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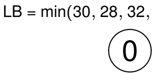

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LB = min(30, 28, 32, 30,

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$$LB = \min(30, 28, 32, 30, 23) = 23$$

0

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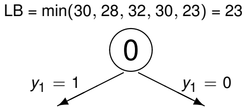

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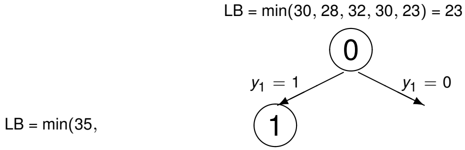

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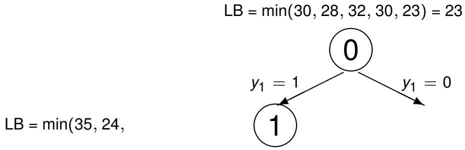

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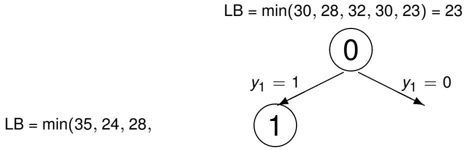

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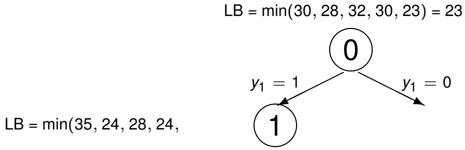

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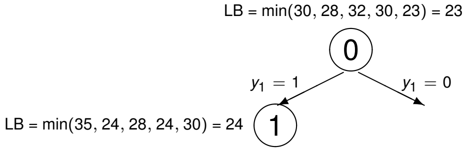

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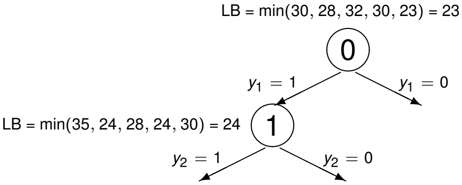

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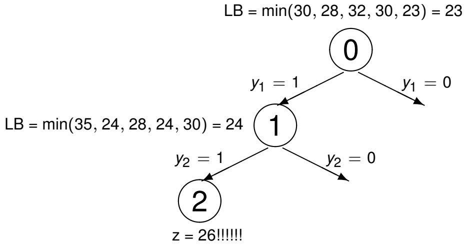

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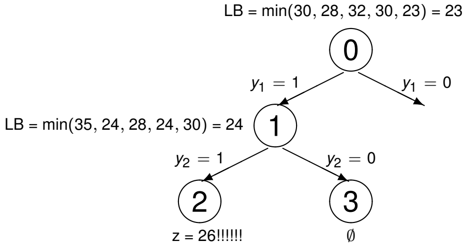

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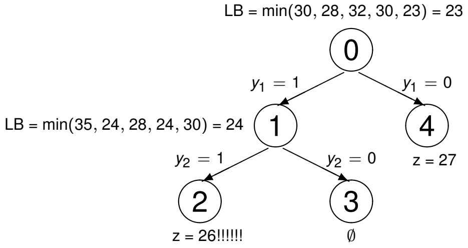

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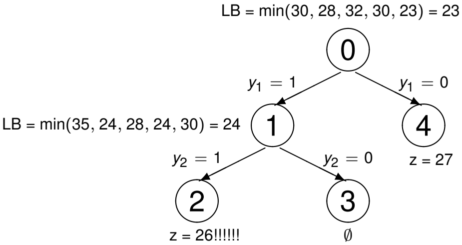

Still not a valid LB!

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MILP relaxation

$$
\begin{aligned}
&\min \gamma \\
f(x^k, y^k) + \nabla f(x^k, y^k)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} &\leq \gamma \qquad \forall k \\
g_i(x^k, y^k) + \nabla g_i(x^k, y^k)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} &\leq 0 \qquad \forall k \, \forall i \in I^k \\
x &\in X \\
y &\in Y.
\end{aligned}
$$

where $I^k \subseteq \{1, 2, \ldots, m\}$.

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where $I^k \subseteq \{1,2,\ldots,m\}$. Two "classical" choices:

- $I^k = \{1,2,\ldots,m\}$

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where $I^k \subseteq \{1, 2, \ldots, m\}$. Two "classical" choices:

- $I^k = \{1, 2, \ldots, m\}$
- $I^k = \{i \mid g(x^k, y^k) > 0, 1 \leq i \leq m\}$

Outer Approximation and nonconvex MINLPs

Several methods for convex MINLPs use **Outer Approximation** cuts (Duran and Grossman, 1986) which are not exact for nonconvex MINLPs.

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$$g(x,y) \le 0 \quad \rightarrow \quad g_i(x^k,y^k) + \nabla g_i(x^k,y^k)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} \le 0$$

where $\nabla g(x^k,y^k)$ is the Jacobian of $g(x,y)$ evaluated at the point (x^k,y^k).

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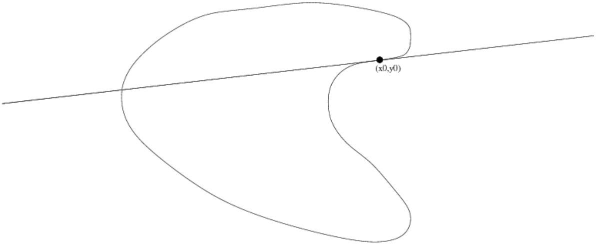

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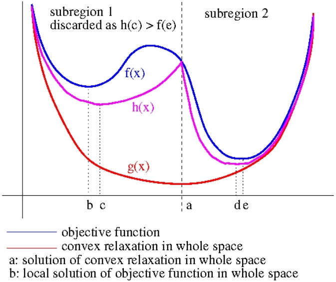
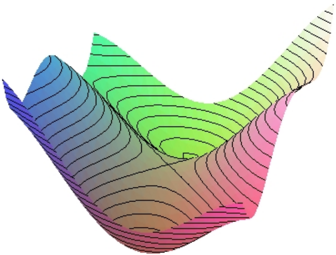
Global Optimization methods



Exact

- "Exact" in continuous space: ε-approximate (*find solution within pre-determined* ε *distance from optimum in obj. fun. value*)
- For some problems, finite convergence to optimum ($\varepsilon = 0$)



Heuristic

- Find solution with probability 1 in infinite time

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Spatial Branch-and-Bound

Falk and Soland (1969) “An algorithm for separable nonconvex programming problems”.

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20 years ago: first general-purpose “exact” algorithms for nonconvex MINLP.

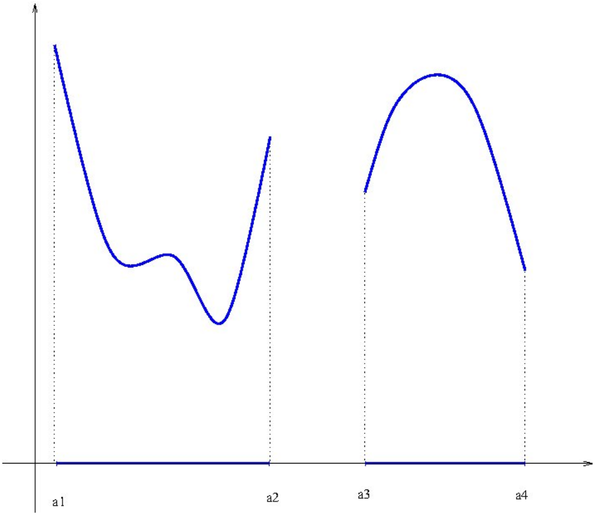
Spatial Branch-and-Bound

Falk and Soland (1969) "An algorithm for separable nonconvex programming problems".

20 years ago: first general-purpose "exact" algorithms for nonconvex MINLP.

- Tree-like search
- Explores search space exhaustively but implicitly
- Builds a sequence of decreasing upper bounds and increasing lower bounds to the global optimum
- Exponential worst-case
- Like BB for MILP, but may branch on continuous vars
  Done whenever one is involved in a nonconvex term

Spatial B&B: Example



Original problem P

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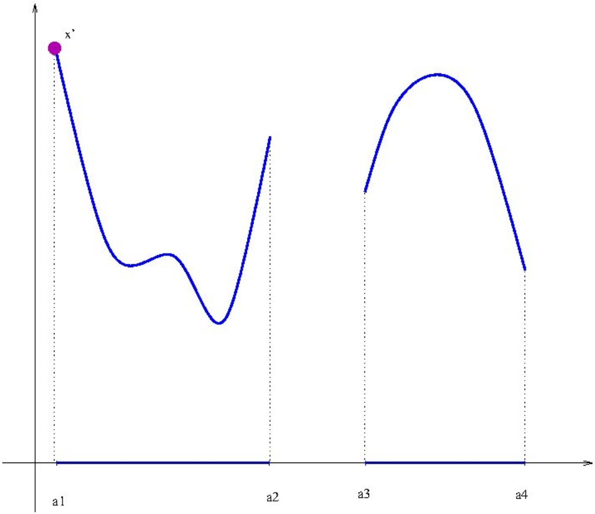

Starting point x'

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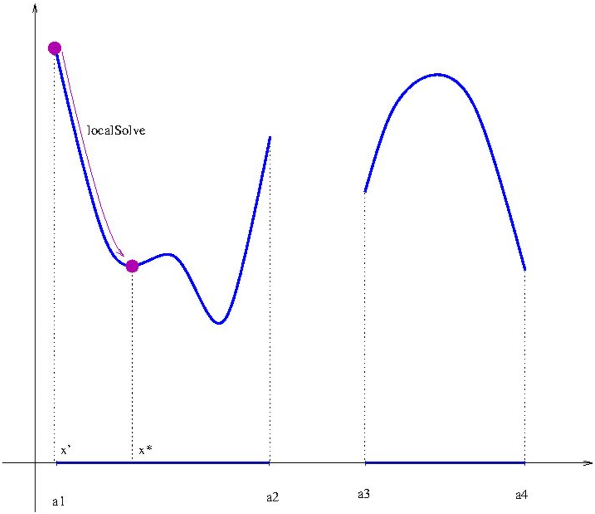

Local (upper bounding) solution x^*

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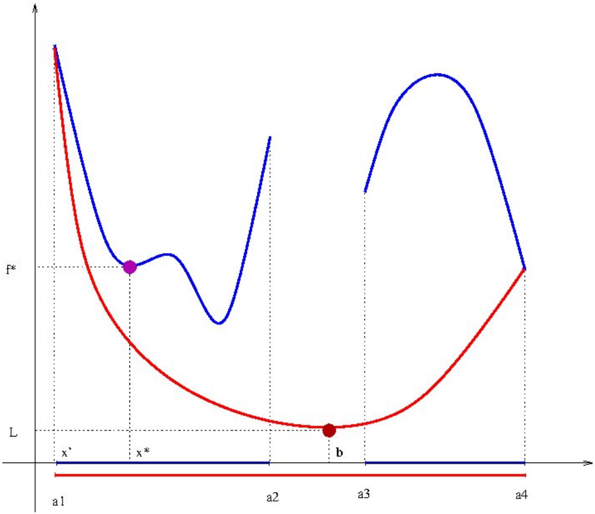
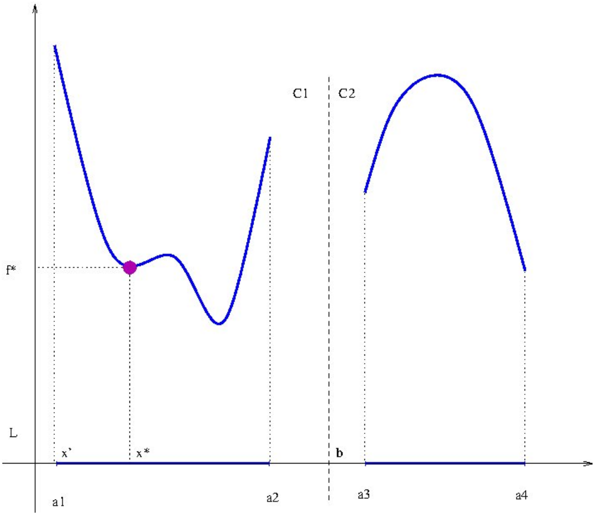


Convex relaxation (lower) bound $\bar{f}$ *with* $|f^* - \bar{f}| > \varepsilon$

Spatial B&B: Example



Branch at $x = \bar{x}$ *into* C_1, C_2

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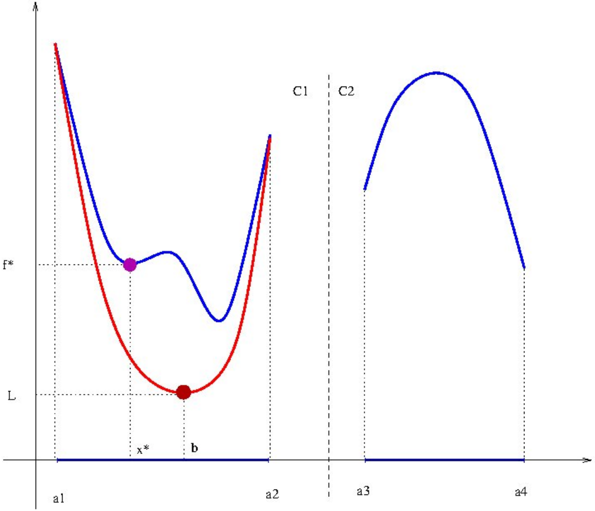

Convex relaxation on C_1: lower bounding solution $\bar{x}$

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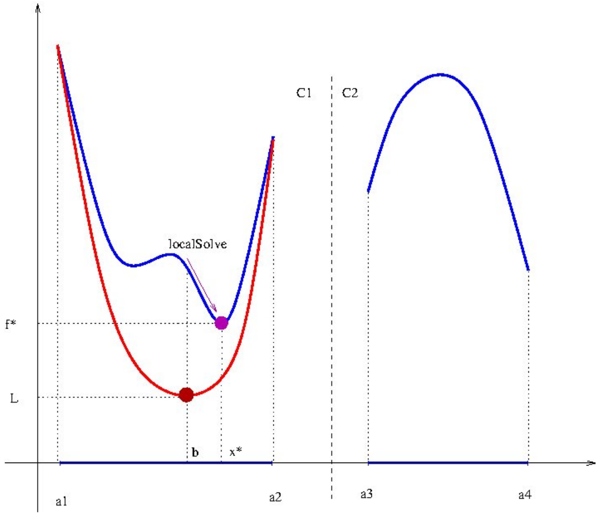
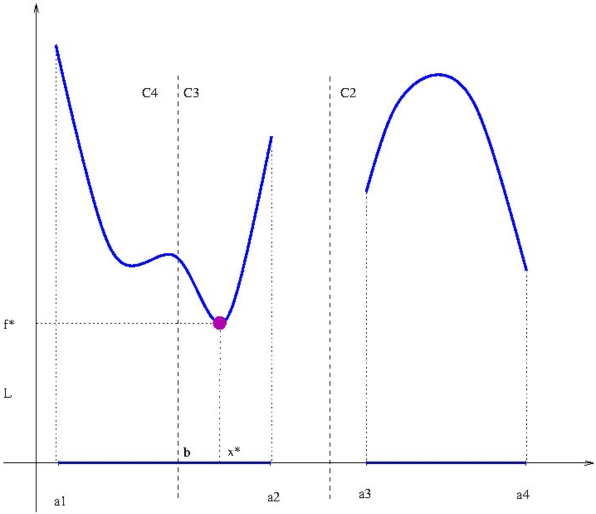


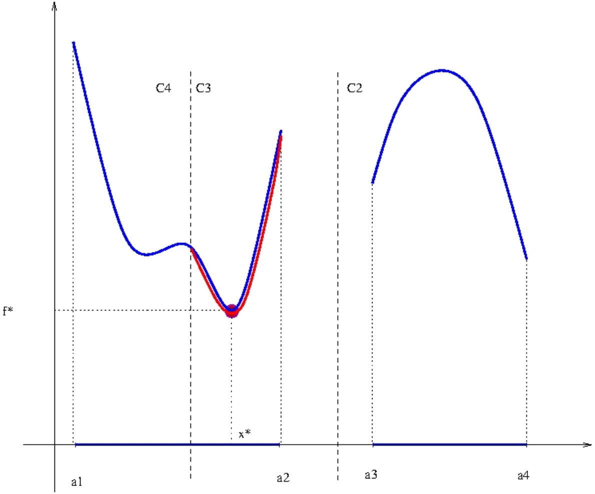
localSolve. *from $\bar{x}$: new upper bounding solution x^**

Spatial B&B: Example



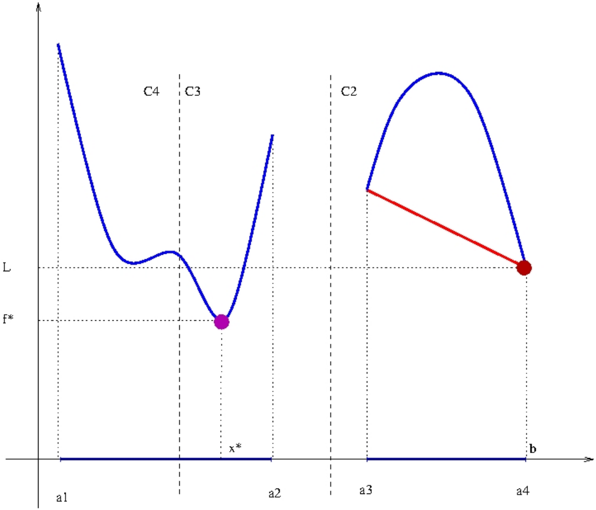
$$|f^* - \bar{f}| > \varepsilon\text{: } \textit{branch at } x = \bar{x}$$

Spatial B&B: Example



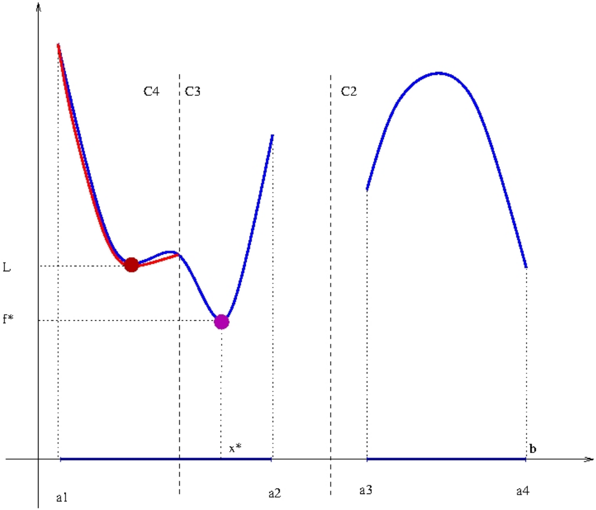
Repeat on C_3: get $\bar{x} = x^*$ and $|f^* - \bar{f}| < \varepsilon$, no more branching

Spatial B&B: Example



Repeat on C_2: $\bar{f} > f^*$ (can't improve x^* in C_2)

Spatial B&B: Example



Repeat on C_4: $\bar{f} > f^*$ *(can't improve* x^* *in* C_4*)*

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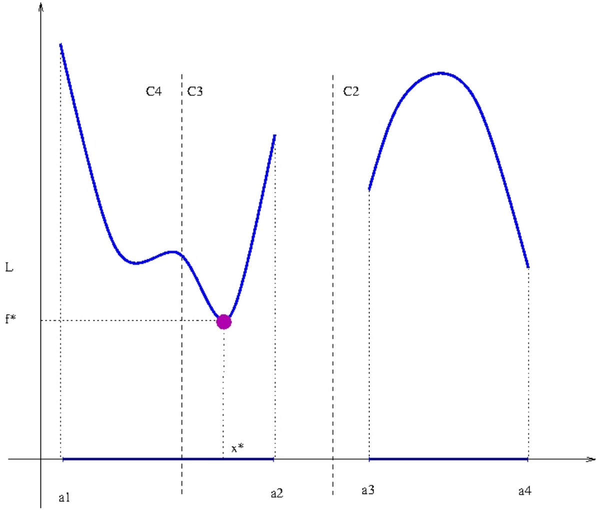

No more subproblems left, return x^* and terminate

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Spatial B&B: Pruning

1. P was branched into C_1, C_2
2. C_1 was branched into C_3, C_4
3. C_3 was **pruned by optimality**
   ($x^* \in \mathcal{G}(C_3)$ was found)
4. C_2, C_4 were **pruned by bound**
   (lower bound for C_2 worse than f^*)
5. No more nodes: whole space explored, $x^* \in \mathcal{G}(P)$

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- Search generates a tree
- Suproblems are nodes
- Nodes can be pruned by optimality, bound or **infeasibility** (when subproblem is infeasible)
- Otherwise, they are branched

Spatial B&B: What is missing?

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1. For arbitrary C, checking if it is feasible is **undecidable**

Spatial B&B: What is missing?

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Spatial B&B: Upper bounds

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Upper bounds: x^* can only decrease

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- Specifically, use *feasible local optima*

Spatial B&B: Lower bounds

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Lower bounds: increase over $\supset$-chains

Spatial B&B: Lower bounds

Lower bounds: increase over $\supset$-chains

- Let R_P be a relaxation of P such that:
  1. R_P also involves the decision variables of P (*and perhaps some others*)

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Lower bounds: increase over $\supset$-chains

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  3. if I, I' are two ranges

     $I \supseteq I' \rightarrow \min R_P[I] \leq \min R_P[I']$

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Lower bounds: increase over $\supset$-chains

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  3. if I, I' are two ranges
     $I \supseteq I' \to \min R_P[I] \leq \min R_P[I']$
  4. For any subproblem C of P,
     finding $x \in \mathcal{G}(R_C)$ or showing $\mathcal{F}(R_C) = \varnothing$ is efficient
     Specifically, $\bar{x} = \text{localSolve}(R_C) \in \mathcal{G}(R_C)$

Spatial B&B: A decidable feasibility test

- Processing C when it's infeasible will make sBB slower but not incorrect

Spatial B&B: A decidable feasibility test

- Processing C when it's infeasible will make sBB slower but not incorrect
- $\Rightarrow$ sBB still works if we simply **never discard a potentially feasible** C

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  - Otherwise, we simply don't know, and we shall process it
- **Thm**: If R_C is infeasible then C is infeasible

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  - Otherwise, we simply don't know, and we shall process it
- **Thm**: If R_C is infeasible then C is infeasible
- **Proof**: $\varnothing = \mathcal{F}(R_C) \supseteq \mathcal{F}(C) = \varnothing$

Tricks

To make an sBB work efficiently, you need further tricks

Expression trees

Representation of objective f and constraints g

Encode mathematical expressions in trees or DAGs

E.g. $x_1^2 + x_1 x_2$:

Expression trees

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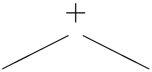

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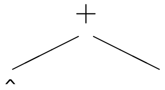

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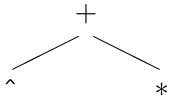

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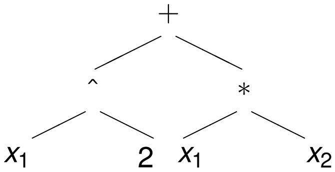

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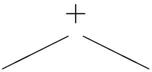

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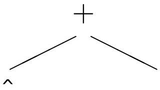

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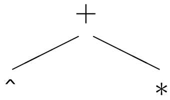

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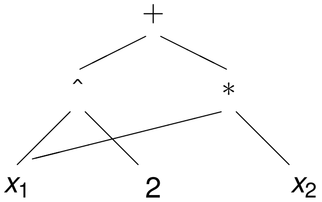
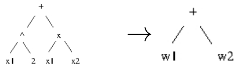


Standard form

- Identify all nonlinear terms $x_i \otimes x_j$, replace them with a linearizing variable w_{ij}
- Add a *defining constraint* $w_{ij} = x_i \otimes x_j$ to the formulation
- Standard form:

$$\left.\begin{array}{rrcl} \min & c^\top(x,w) & & \\ \text{s.t.} & A(x,w) & \begin{smallmatrix}\le\\=\\\ge\end{smallmatrix} & b \\ & w_{ij} & = & x_i \otimes_{ij} x_j \text{ for suitable } i,j \\ & \text{bounds} & \& & \text{integrality constraints} \end{array}\right\}$$

- $x_1^2 + x_1 x_2 \Rightarrow \begin{cases} w_{11} + w_{12} \\ w_{11} = x_1^2 \\ w_{12} = x_1 x_2 \end{cases}$:



Convex relaxation

- *Standard form*: all nonlinearities in defining constraints
- Each defining constraint $w_{ij} = x_i \otimes x_j$ is replaced by two convex inequalities:

$$
\begin{aligned}
w_{ij} &\leq \text{overestimator}(x_i \otimes x_j) \\
w_{ij} &\geq \text{underestimator}(x_i \otimes x_j)
\end{aligned}
$$

- Convex relaxation is not the tightest possible, but it can be constructed automatically

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Summary

ORIGINAL MINLP
$\min_x f(x)$ $g(x) \leq 0$ $x^L \leq x \leq x^U$

Some variables may be integral

STANDARD FORM
$\min c^\top(x,w)$ $A(x,w) = b$ $w_i = \phi_i(x,w)\ \forall i$ $w^L \leq w \leq w^U$

- Easier to perform symbolic algorithms
- Linearizes nonlinear terms
- Adds linearizing variables and defining constraints

CONVEX RELAXATION
$\min c^\top(x,w)$ $A(x,w) = b$ relax def constr $w_i\ \forall i$ $w^L \leq w \leq w^U$

Each defining constraint replaced by convex under- and concave over-estimators

Variable ranges

- Crucial property for sBB convergence: **convex relaxation tightens as variable range widths decrease**
- convex/concave under/over-estimator constraints are (convex) functions of x^L, x^U
- it makes sense to **tighten** x^L, x^U at the sBB root node (trading off speed for efficiency) and at each other node (trading off efficiency for speed)

Bounds Tightening

- In sBB we need to tighten variable bounds at each node
- Example:
  - Optimization Based Bounds Tightening (OBBT)

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- In sBB we need to tighten variable bounds at each node
- Example:
  - Optimization Based Bounds Tightening (OBBT)
- OBBT:
  for each variable x in P compute
  - $\underline{x} = \min\{x \mid \text{conv. rel. constr.}\}$
  - $\overline{x} = \max\{x \mid \text{conv. rel. constr.}\}$

  Set $\underline{x} \leq x \leq \overline{x}$

References

- Falk, Soland, " An algorithm for separable nonconvex programming problems" , Manag. Sci. 1969.
- Horst, Tuy, " Global Optimization" , Springer 1990.
- Ryoo, Sahinidis, " Global optimization of nonconvex NLPs and MINLPs with applications in process design" , Comp. Chem. Eng. 1995.
- Adjiman, Floudas et al., " A global optimization method, αBB, for general twice-differentiable nonconvex NLPs" , Comp. Chem. Eng. 1998.
- Smith, Pantelides, " A symbolic reformulation/spatial branch-and-bound algorithm for the global optimisation of nonconvex MINLPs" , Comp. Chem. Eng. 1999.
- Nowak, " Relaxation and decomposition methods for Mixed Integer Nonlinear Programming" , Birkhäuser, 2005.
- Belotti, Liberti et al., " Branching and bounds tightening techniques for nonconvex MINLP" , Opt. Meth. Softw., 2009.
- Vigerske, PhD Thesis: " Decomposition of Multistage Stochastic Programs and a Constraint Integer Programming Approach to Mixed-Integer Nonlinear Programming" , Humboldt-University Berlin, 2013.

How far can we get?

Example: Box-constrained polynomial optimization.

$$\begin{array}{llll} \min & f(x,y) & & \\ & x_i \in & [l_i, u_i] & \forall i \in \{1,\ldots,p\} \\ & y_i \in & \{l_i,\ldots,u_i\} & \forall i \in \{1,\ldots,q\}\ , \end{array}$$

where f is an arbitrary polynomial of degree $d \in \mathbb{N}$.

Let us define $n = p + q$ and $m =$ the number of monomials of f.

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How far can we get?

Example: Box-constrained polynomial optimization.

n	*m*	**scip**	**couenne**	**baron**
10	10	0.33 (10)	0.08 (10)	1.34 (10)
10	20	1.43 (10)	0.44 (10)	0.89 (10)
10	30	22.05 (10)	0.81 (10)	3.08 (10)
10	40	182.85 (9)	4.56 (10)	26.89 (10)
10	50	774.88 (9)	16.26 (10)	54.59 (10)
10	60	447.96 (5)	22.29 (10)	53.64 (10)
10	70	679.35 (4)	11.25 (10)	126.53 (10)
10	80	1574.58 (1)	74.86 (10)	577.44 (10)
10	90	3474.68 (1)	72.41 (10)	263.55 (10)
10	100	***	51.78 (9)	567.50 (10)
15	15	0.19 (10)	0.15 (10)	0.44 (10)
15	30	112.57 (10)	1.98 (10)	6.67 (10)
15	45	318.58 (7)	10.49 (9)	125.61 (10)
15	60	879.22 (2)	117.01 (10)	556.31 (10)
15	75	***	318.52 (10)	967.85 (8)
15	90	***	301.10 (6)	1440.39 (3)
15	105	***	495.67 (6)	***
15	120	***	586.83 (6)	952.72 (1)
15	135	***	1673.22 (4)	***
15	150	***	1614.30 (2)	***

Table: Results for bounds [-10,10], mixed-integer variables.

How far can we get?

Example: Box-constrained polynomial optimization.

n	m	**scip**	**couenne**	**baron**
20	20	0.51 (10)	0.33 (10)	0.86 (10)
20	40	707.09 (8)	12.50 (10)	113.98 (10)
20	60	***	383.20 (10)	1842.67 (6)
20	80	***	1141.59 (7)	***
20	100	***	1110.76 (2)	***
20	120	***	1984.69 (1)	***
20	140	***	***	***
20	160	***	1223.26 (1)	***
20	180	***	***	***
20	200	***	***	***
25	25	2.15 (10)	0.80 (10)	2.92 (10)
25	50	1233.17 (1)	51.20 (10)	606.13 (9)
25	75	***	1237.38 (6)	3378.23 (1)
25	100	***	1167.83 (1)	***
25	125	***	***	***
25	150	***	***	***
25	175	***	***	***
25	200	***	***	***
25	225	***	***	***
25	250	***	***	***

Table: Results for bounds [-10,10], mixed-integer variables.

Example...

$$
\begin{aligned}
\min(((x_0 * x_2) * (-0.47373 * y_7)) + ((0.218418 * y_7) * (x_4 * (x_0 * x_1))) + \\
((0.843784 * y_6) * (x_3 * (x_0 * x_2))) + (0.914311 * (y_9 * (y_5 * y_6))) + \\
(x_2 * (-0.620254 * (y_5 * y_8))) + ((x_0 * x_4) * (0.103064 * (y_7 * y_8))) + \\
(x_2 * (-0.300792 * (y_9 * (y_5 * y_7)))) + ((-0.788548 * y_7) * (x_1 * (x_2^2))) + \\
((x_1 * x_2) * (-0.185507 * (y_6^2))) + (x_1 * (0.428212 * (y_9^2))))
\end{aligned}
$$

$$
\begin{aligned}
x_0 &\in [0, 1] \\
x_1 &\in [0, 1] \\
x_2 &\in [0, 1] \\
x_3 &\in [0, 1] \\
x_4 &\in [0, 1] \\
y_5 &\text{ binary} \\
y_6 &\text{ binary} \\
y_7 &\text{ binary} \\
y_8 &\text{ binary} \\
y_9 &\text{ binary}
\end{aligned}
$$

Problem size before reformulation: 10 variables (5 integer), 0 constraints.

Example...

$$
\begin{aligned}
& \min w_{34} \\
x_0 &\in [0,1] \\
x_1 &\in [0,1] \\
x_2 &\in [0,1] \\
x_3 &\in [0,1] \\
x_4 &\in [0,1] \\
& y_5 \text{ binary} \\
& y_6 \text{ binary} \\
& y_7 \text{ binary} \\
& y_8 \text{ binary} \\
& y_9 \text{ binary} \\
w_{10} &:= (x_0 * x_2) \in [0,1] \\
w_{11} &:= (y_7 * w_{10}) \in [0,1] \\
w_{12} &:= (x_0 * x_1) \in [0,1] \\
w_{13} &:= (x_4 * w_{12}) \in [0,1] \\
w_{14} &:= (y_7 * w_{13}) \in [0,1] \\
w_{15} &:= (x_3 * w_{10}) \in [0,1] \\
w_{16} &:= (y_6 * w_{15}) \in [0,1] \\
z_{17} &:= (y_5 * y_6) \text{ binary} \\
z_{18} &:= (y_9 * z_{17}) \text{ binary} \\
w_{19} &:= (x_2 * y_5) \in [0,1] \\
& \ldots
\end{aligned}
$$

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Example...

$$
\begin{aligned}
&\min w_34 \\
&\dots \\
w_{20} &:= (y_8 * w_{19}) \in [0,1] \\
w_{21} &:= (x_0 * x_4) \in [0,1] \\
w_{22} &:= (y_7 * w_{21}) \in [0,1] \\
w_{23} &:= (y_8 * w_{22}) \in [0,1] \\
w_{24} &:= (y_7 * w_{19}) \in [0,1] \\
w_{25} &:= (y_9 * w_{24}) \in [0,1] \\
w_{26} &:= (x_2^2) \in [0,1] \\
w_{27} &:= (x_1 * y_7) \in [0,1] \\
w_{28} &:= (w_{26} * w_{27}) \in [0,1] \\
z_{29} &:= (y_6^2) \in \text{binary} \\
w_{30} &:= (x_1 * x_2) \in [0,1] \\
w_{31} &:= (z_{29} * w_{30}) \in [0,1] \\
z_{32} &:= (y_9^2) \in \text{binary} \\
w_{33} &:= (x_1 * z_{32}) \in [0,1] \\
w_{34} &:= (-0.47373 * w_{11} + 0.218418 * w_{14} + 0.843784 * w_{16} + \\
&0.914311 * z_18 - 0.620254 * w_{20} + 0.103064 * w_{23} - 0.300792 * w_{25} - \\
&0.788548 * w_{28} - 0.185507 * w_{31} + 0.428212 * w_{33}) \in [-2.36883, 2.50779]
\end{aligned}
$$

Problem size after reformulation: 35 variables (9 integer), 0 constraints.

Outline

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Nonconvex MINLP solvers

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Need to evaluate function, its first and its second derivative at (x^*, y^*).

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Possible source of errors!

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Need to evaluate function, its first and its second derivative at (x^*, y^*).
Possible source of errors! Solution? Modeling Languages!

Modeling Languages

Modeling languages, e.g., `AMPL` and `GAMS`.

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Modeling Languages

Modeling languages, e.g., AMPL and GAMS.
Example:

```
param pi := 3.142;
param N;
set VARS ordered := {1..N};
param Umax default 100;
param U {j in VARS};
param a {j in VARS};
param b {j in VARS};
param c {j in VARS};
param d {j in VARS};
param w{VARS};
param C;
```


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Neos
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Smart Grids

Modeling Languages

Modeling languages, e.g., AMPL and GAMS.
Example:

```
param pi := 3.142;
param N;
set VARS ordered := {1..N};
param Umax default 100;
param U {j in VARS};
param a {j in VARS};
param b {j in VARS};
param c {j in VARS};
param d {j in VARS};
param w{VARS};
param C;
var y {j in VARS} >= 0, <= U[j], integer;
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subject to KP_constraint:  sum{j in VARS} w[j]*y[j] <= C;
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Neos

NEOS: `http://www.neos-server.org/neos/`.

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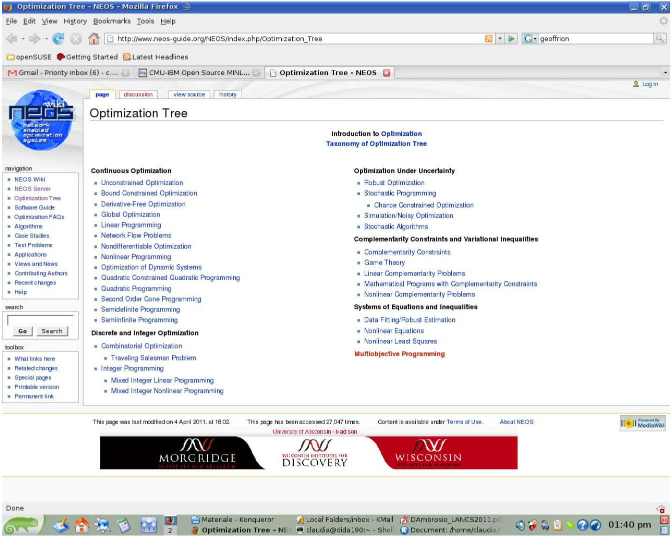

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- MINLPlib: 270 instances
  `http://www.gamsworld.org/minlp/minlplib.htm`

Smart Grids?

Production, storage, and distribution of electricity in a smart micro grid.

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S. Toubaline, P.-L. Poirion, C. D'Ambrosio, L. Liberti. Observing the state of a smart grid using bilevel programming. **COCOA 2015** (accepted).

Thank you!

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