First-passage time analysis for Markovian deteriorating model

Gejza Dohnal



# Successive events and Energy Networks
# Markovian deteriorating model

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Modeling Smart Grids - A new Challenge for Stochastics and Optimization

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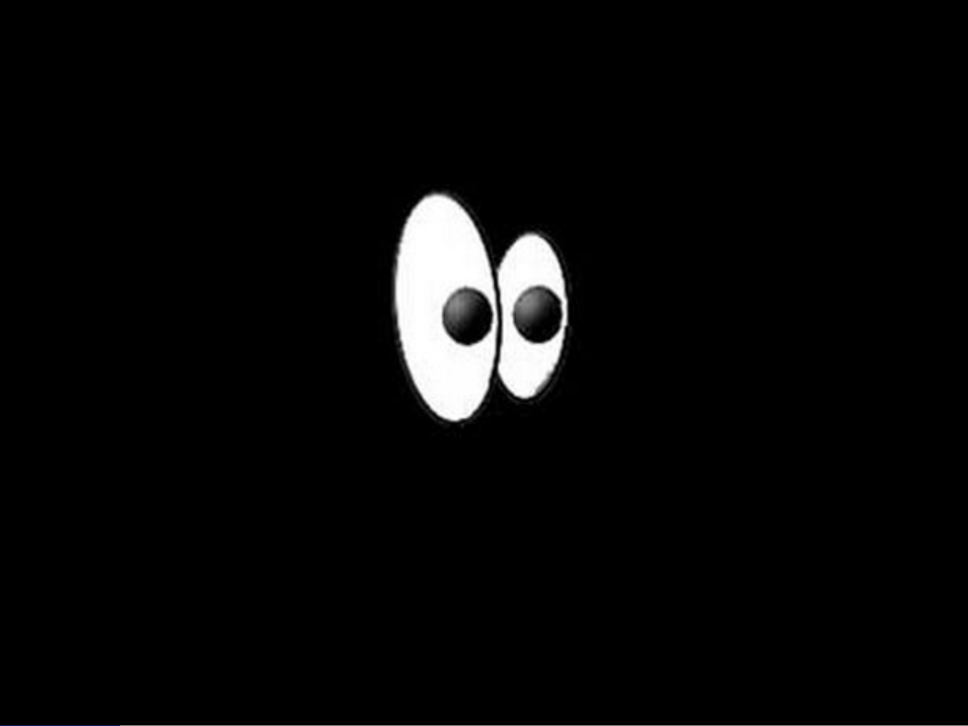
# Introduction

**August 14th, 2003.**
**That day in the northeastern US the temperature reached 90°C, which is more than the average of 80°F but less than the maximum 100°F. In the network, power flows dominated from the south and west to the north (Michigan) and East (New York). Export from the south and west of the US directed mainly to northern Ohio to Michigan and Ontario, Canada. The fault occured in northern Ohio due to a short circuit on contact wires overhead lines of 345 kV with a tree. Overloading caused the cascading spread of successive faults, gradual shutdown of transmission lines due to protection actions, formation of island traffic and then blackout in large parts of the northeast United States and southeast Canada. The total loss in sixteen hours accounted for 61,800 MW. The event hit 265 plants with 508 blocks. Without electricity remained 50 million inhabitants.**



# Introduction

**Failure** $\times$ **Disaster**

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| independent recurrence | | growing spread |
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**Model** $\longrightarrow$ **Prediction** $\longrightarrow$ **Prevention**

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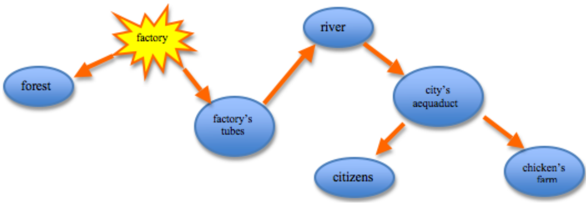
**Model $\longrightarrow$ Prediction $\longrightarrow$ Prevention**

Preventive Maintenance Policy | Disaster Recovery Plan

# Model


factory
forest
factory's tubes
river
city's aequaduct
citizens
chicken's farm

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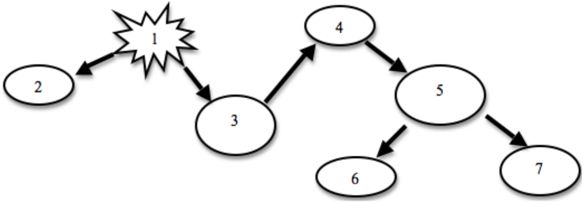

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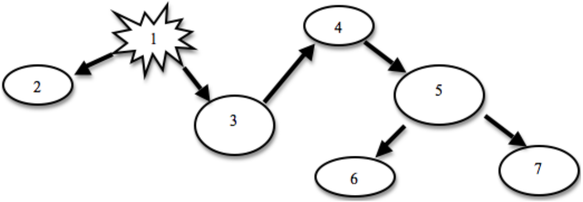


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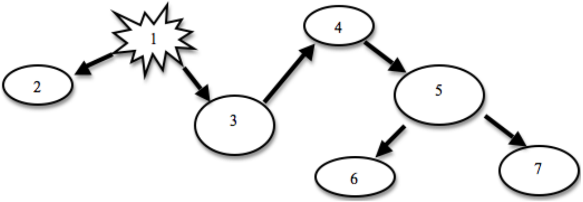


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**(i)** at the beginning (at the time 0), the system is in the state $\omega^0 = (0, 0, \ldots, 0)$,

**(ii)** the process starts by deterioration of an object $i$ with probability $\pi_i, i = 1, 2, \ldots, n$,

**(iii)** when at the time $t$ an object $i$ was affected, there was a random time period $\tau$ after which the event moved onto some of the unaffected objects,

**(iv)** states of the system in time $t$ create a stochastic process $\{X(t), t \geq 0\}$ in continuous time. Values of this process lie within the set $\Omega = \{0, 1\}^n$

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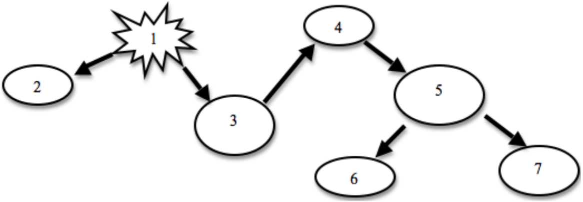


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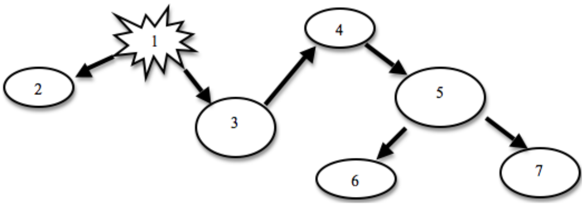


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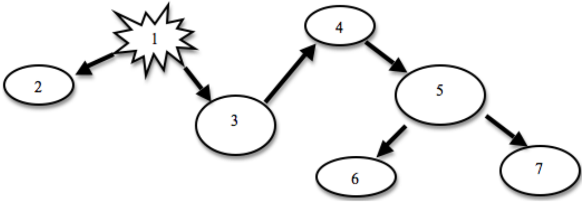


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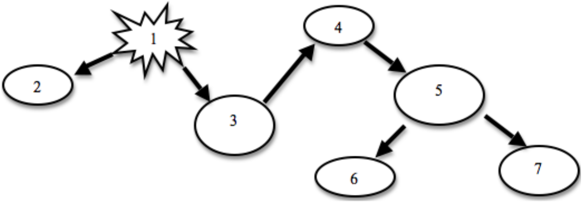
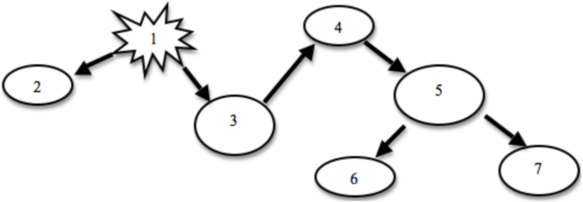



**Example ($n = 7 \quad \pi = (1, 0, 0, 0, 0, 0, 0)$)**

From all $2^7$ possible states only the following 14 are admissible:

(1,0,0,0,0,0,0), (1,1,0,0,0,0,0), (1,0,1,0,0,0,0), (1,1,1,0,0,0,0), (1,0,1,1,0,0,0), (1,1,1,1,0,0,0), (1,0,1,1,1,0,0), (1,1,1,1,1,0,0), (1,0, 1,1,1,1,0), (1,0,1,1,1,0,1),(1,1,1,1,1,1,0), (1,1,1,1,1,0,1), (1,0,1,1,1,1,1), (1,1,1,1,1,1,1).

# Model 1




## Model 1 – One-way Model

Let us consider the following supplementary assumptions:

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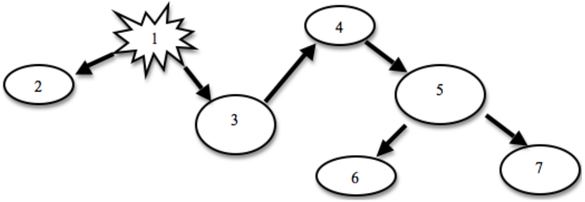


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(v) an event can only affect one object in one moment,

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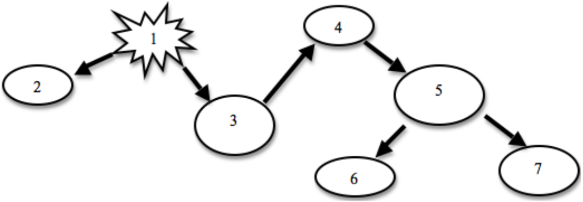


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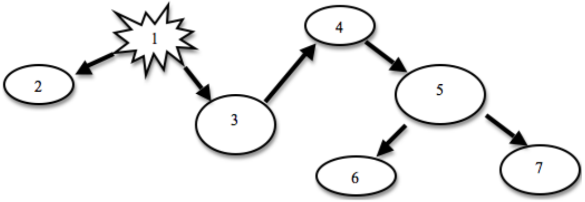


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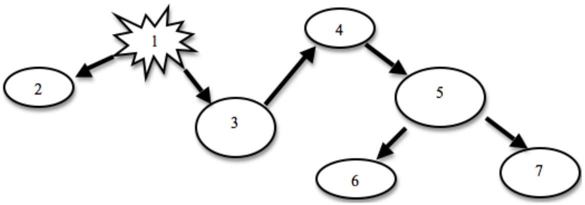


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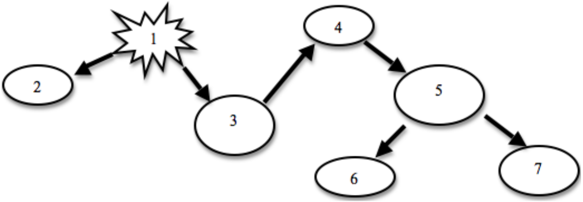


Let us denote:

- $|\omega| = \omega_1 + \omega_2 + \cdots + \omega_n$ the number of objects, on which a disastrous event holds when system is in the state $\omega$ (*size of deterioration* of the system)

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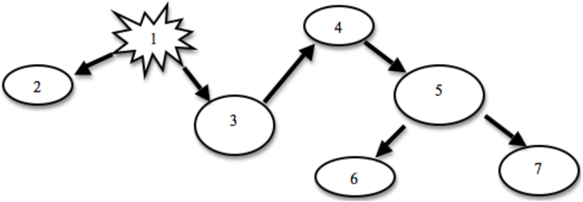


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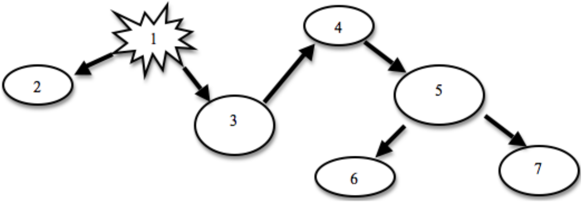


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$$\Omega = \bigcup_{j=1}^{n} \Omega_j, \quad \Omega_i \cap \Omega_j = \emptyset \text{ for all } i \neq j$$

# Model 1 – One-way Model

## Proposition

*Let us order states of the system in ascending order by the size of deterioration. Then, under assumptions (i)-(vii), the transition intensity matrix Q of the process $\{X(t), t \geq 0\}$ is upper triangular of the size $2^n \times 2^n$. The matrix Q can be written in the block-form as*

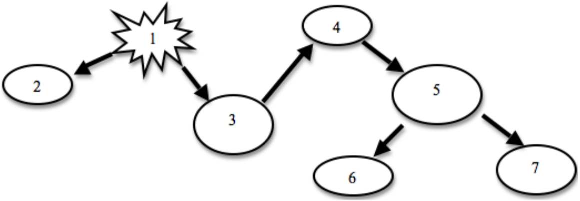
$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{D}_{0,0} & \boldsymbol{P}_{0,1} & \boldsymbol{O}_{0,2} & \boldsymbol{O}_{0,3} & \cdots & \boldsymbol{0} \\ \boldsymbol{O}_{1,0} & \boldsymbol{D}_{1,1} & \boldsymbol{P}_{1,2} & \boldsymbol{O}_{1,3} & \cdots & \boldsymbol{O}_{1,n} \\ \boldsymbol{O}_{2,0} & \boldsymbol{O}_{2,1} & \boldsymbol{D}_{2,2} & \boldsymbol{P}_{2,3} & \cdots & \boldsymbol{O}_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{O}_{n,1} & \boldsymbol{O}_{n,2} & \boldsymbol{O}_{n,3} & \cdots & \boldsymbol{0} \end{pmatrix}$$

*where $P_{i,j}$ is a rectangular matrix of size $\binom{n}{i} \times \binom{n}{j}$, the symbol $O_{ij}$ denotes a null matrix and $D_{ii}$ are square diagonal matrices for $i = 1, \ldots, n$. Moreover,*

$$-D_{i,i} = e.P'_{i,i+1}.I_{i,i}$$

*where $I_{i,i}$ is the unit matrix of size i, e is the row vector of $\binom{n}{j}$ ones.*

# Model 2




## Model 2 – Repairable Model

Let us omit the assumption (vi):

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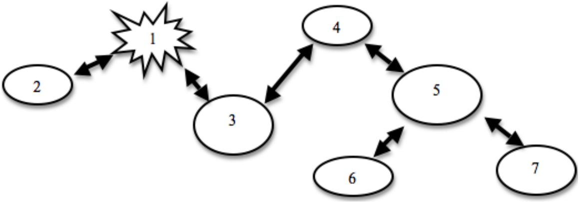


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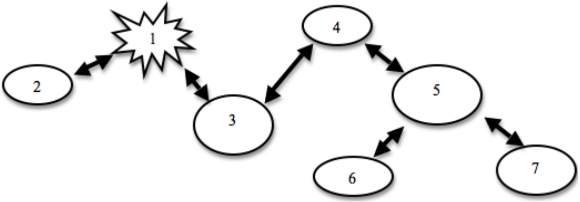


The process $X(t)$ can be considered as a random walk between sets $\Omega_0, \ldots, \Omega_n$. When the process in a state $\omega \in \Omega_j,\ 0 < j < n$, the only transitions to some states in $\Omega_{j-1}$ or $\Omega_{j+1}$ are allowed, whereas $\Omega_0 = \{\omega^0\}$ and $\Omega_N = \{\omega^N\}$ are reflection states. In the model, all states are transient. The process can be understood as the process of event spreading in the system with repair.

# Model 2 – Repairable Model

## Proposition

*In the model 2, the transition intensity matrix Q of the process $\{X(t), t \geq 0\}$ has the following block-form*

$$Q = \begin{pmatrix} \boldsymbol{D_{0,0}} & \boldsymbol{P_{0,1}} & \boldsymbol{O_{0,2}} & \boldsymbol{O_{0,3}} & \cdots & \boldsymbol{0} \\ \boldsymbol{R_{1,0}} & \boldsymbol{D_{1,1}} & \boldsymbol{P_{1,2}} & \boldsymbol{O_{1,3}} & \cdots & \boldsymbol{O_{1,n}} \\ \boldsymbol{O_{2,0}} & \boldsymbol{R_{2,1}} & \boldsymbol{D_{2,2}} & \boldsymbol{P_{2,3}} & \cdots & \boldsymbol{O_{2,n}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{O_{n,1}} & \boldsymbol{O_{n,2}} & \boldsymbol{O_{n,3}} & \cdots & \boldsymbol{D_{n,n}} \end{pmatrix}$$

*where $P_{i,j}$ is a rectangular matrix of size $\binom{n}{i} \times \binom{n}{j}$, $R_{j,i}$ is a rectangular matrix of size $\binom{n}{j} \times \binom{n}{i}$, the symbol $O_{ij}$ denotes a null matrix and $D_{ii}$ are square diagonal matrices for $i = 1, \ldots, n$. Moreover,*

$$-D_{i,i} = (e^i.R'_{i-1,i} + e^{i+1}.P'_{i,i+1}).I_{i,i}$$

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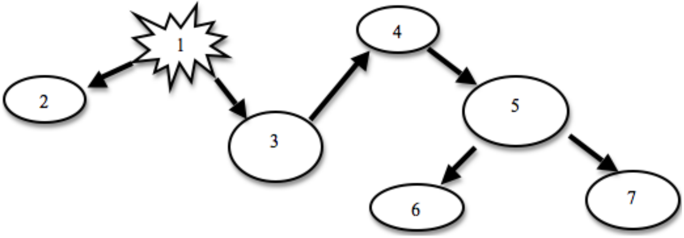
# Example

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The initial probability vector is equal to $\pi = (1, 0, 0, 0, 0, 0, 0)$.

# Example

The transient intensity matrix $S$ has the form

$$\begin{pmatrix}
u & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v & d & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -f & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & w & g & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -i & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x & j & k & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -l & 0 & 0 & l & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & m & 0 & n & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & p & q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r & 0 & 0 & r \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s & 0 & s \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

# Example





$$Se = 0 \;\Rightarrow\; u = -a - b,\; v = -d - e,\; w = -g - h,$$
$$x = -j - k,\; y = -m - n,\; z = -p - q.$$

There remains 19 unknown parameters which correspond to conditional transient intensities:

$$\begin{array}{lll}
a = i(1 \to 2|1); & g = i(1 \to 2|1,3,4); & m = i(1 \to 2|1,3,4,5,6); \\
b = i(1 \to 3|1); & h = i(4 \to 5|1,3,4); & n = i(5 \to 7|1,3,4,5,6); \\
c = i(1 \to 3|1,2); & i = i(4 \to 5|1,2,3,4); & p = i(1 \to 2|1,3,4,5,7); \\
d = i(1 \to 2|1,3); & j = i(1 \to 2|1,3,4,5); & q = i(5 \to 6|1,3,4,5,7); \\
e = i(3 \to 4|1,2); & k = i(5 \to 6|1,3,4,5); & r = i(5 \to 7|1,2,3,4,5,6); \\
f = i(3 \to 4|1,3); & l = i(5 \to 6|1,2,3,4,5); & s = i(5 \to 6|1,2,3,4,5,7); \\
 & & t = (1 \to 2|1,3,4,5,6,7).
\end{array}$$

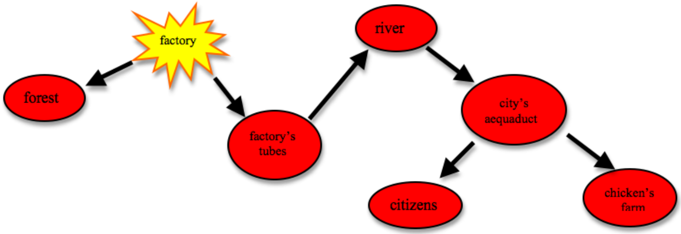
In the case, where transitions between objects are independent of the previous path, we have

$$a = d = g = j = m = p = t,\;\; b = c,\;\; e = f,\;\; h = i,\;\; k = l = q = s,\;\; n = r.$$

The whole system can be reduced to 6 unknown parameters $a, b, e, h, k, n$.

# Lifetime of the system






**One-way Model:** The time of the total deterioration of the system is the time $T$, in which the process $X(t)$ will attach the state $\omega^N$.

During this time, the pursued events will pass through all objects of the system, the state $\omega^N$ is absorbing.

## Proposition

*In the model 1, the time $T$ to the system deterioration is a random variable, which has phase type probability distribution with initial probability vector* $\pi = (\pi_1, \pi_2, \ldots, \pi_n, 0, 0, \ldots, 0)$ *and upper triangular transition intensity matrix*

$$S = \begin{pmatrix} D_{1,1} & P_{1,2} & O_{1,3} & \cdots & O_{1,n-1} \\ O_{2,1} & D_{2,2} & P_{2,3} & \cdots & O_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{n-1,1} & O_{n-1,2} & O_{n-1,3} & \cdots & D_{n-1,n-1} \end{pmatrix}.$$

## Proposition

*In the model 1, the time $T$ to the system deterioration is a random variable, which has phase type probability distribution with initial probability vector* $\pi = (\pi_1, \pi_2, \ldots, \pi_n, 0, 0, \ldots, 0)$ *and upper triangular transition intensity matrix*

$$\mathbf{S} = \begin{pmatrix} \mathbf{D}_{1,1} & \mathbf{P}_{1,2} & \mathbf{O}_{1,3} & \cdots & \mathbf{O}_{1,n-1} \\ \mathbf{O}_{2,1} & \mathbf{D}_{2,2} & \mathbf{P}_{2,3} & \cdots & \mathbf{O}_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O}_{n-1,1} & \mathbf{O}_{n-1,2} & \mathbf{O}_{n-1,3} & \cdots & \mathbf{D}_{n-1,n-1} \end{pmatrix}.$$

For PH-distribution, we know all moments

$$E(T^k) = (-1)^k k! \pi S^{-k} e, \quad k \in N.$$

where $e$ is vector of ones.

## Proposition

*In model 2, the time $T$ to the system deterioration is a random variable, which has phase type probability distribution with initial probability vector $\pi = (1, 0, \ldots, 0)$ and transition intensity matrix*

$$S = \begin{pmatrix} D_{0,0} & P_{0,1} & O_{0,2} & \cdots & O_{0,n-1} \\ R_{1,0} & D_{1,1} & P_{1,2} & \cdots & O_{1,n-1} \\ 0_{2,0} & R_{2,1} & D_{2,2} & \cdots & O_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{n-1,0} & O_{n-1,1} & O_{n-1,2} & \cdots & D_{n-1,n-1} \end{pmatrix}.$$

# Lifetime of the system
## Model 2

### Proposition

*In model 2, the time $T$ to the system deterioration is a random variable, which has phase type probability distribution with initial probability vector $\pi = (1, 0, \ldots, 0)$ and transition intensity matrix*

$$\mathbf{S} = \begin{pmatrix} \mathbf{D}_{0,0} & \mathbf{P}_{0,1} & \mathbf{O}_{0,2} & \cdots & \mathbf{O}_{0,n-1} \\ \mathbf{R}_{1,0} & \mathbf{D}_{1,1} & \mathbf{P}_{1,2} & \cdots & \mathbf{O}_{1,n-1} \\ \mathbf{0}_{2,0} & \mathbf{R}_{2,1} & \mathbf{D}_{2,2} & \cdots & \mathbf{O}_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O}_{n-1,0} & \mathbf{O}_{n-1,1} & \mathbf{O}_{n-1,2} & \cdots & \mathbf{D}_{n-1,n-1} \end{pmatrix}.$$

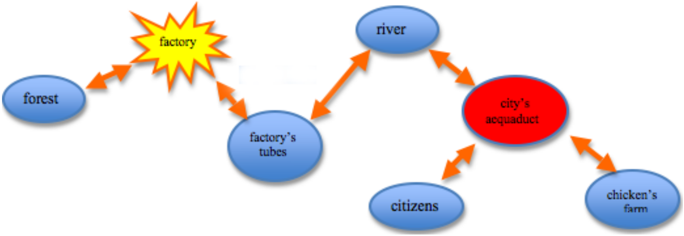
Particularly,

$$E(T) = -\pi S^{-1} e,$$

$$Var(T) = 2\pi S^{-2} e - (\pi S^{-1} e)^2.$$

# First-affect time





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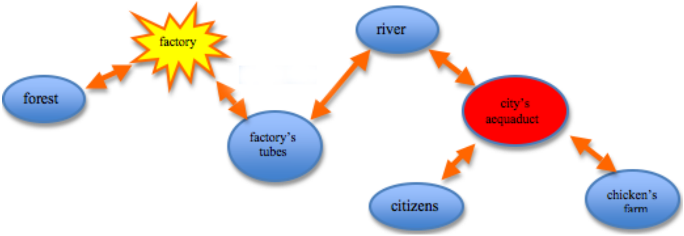
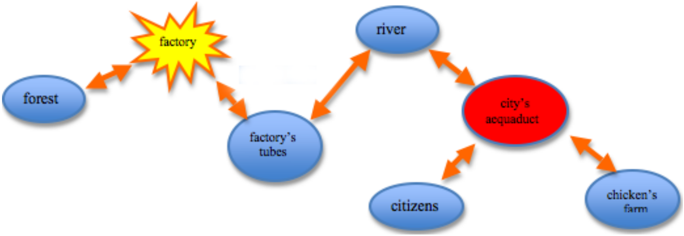



For given $i$ denote

$$S_0^i = \{\omega \in \Omega : \omega_i = 0\}$$

$$S_1^i = \{\omega \in \Omega : \omega_i = 1\}$$

# First-affect time




For given $i$ denote

$S_0^i = \{\omega \in \Omega : \omega_i = 0\}$ – the set of states of the system, in which the $i$-th object is not affected

$S_1^i = \{\omega \in \Omega : \omega_i = 1\}$

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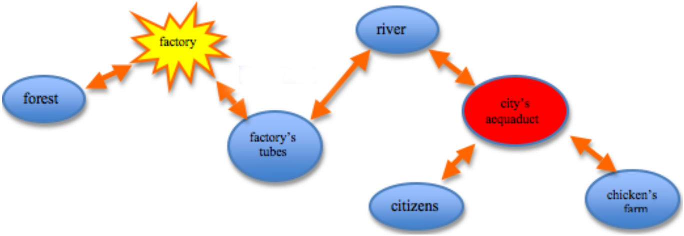


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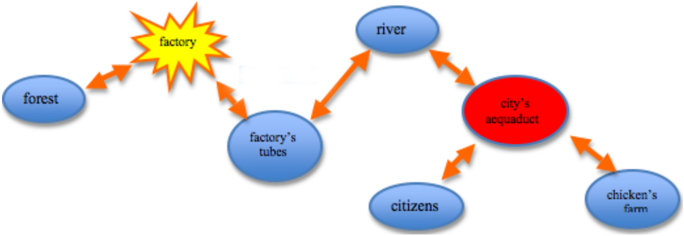


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The sets $S_0^i$ and $S_1^i$ are disjoint and $S_0^i = \Omega - S_1^i$

# First-affect time

## One-way Model

Reordering states in both of these sets in ascending order according to $|\omega|$, we can write the transition intensity matrix in the following block form:

$$\begin{array}{cc} & \begin{array}{cc} S_0^i & S_1^i \end{array} \\ \begin{array}{c} S_0^i \\ S_1^i \end{array} & \begin{pmatrix} A & C \\ O & B \end{pmatrix} \end{array}$$

where

$A$ is upper triangular of the size $(2^{n-1} - 1) \times (2^{n-1} - 1)$,

$B$ is is upper triangular of the size $(2^{n-1} \times 2^{n-1})$,

# First-affect time





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## Repairable Model

After reordering of states we obtain the transition intensity matrix

$$\begin{array}{cc} & \begin{array}{cc} S_0^i & S_1^i \end{array} \\ \begin{array}{c} S_0^i \\ S_1^i \end{array} & \begin{pmatrix} A & C \\ G & B \end{pmatrix} \end{array}$$

where $G$ is not null matrix.

# First-affect time





**Proposition**

*The first affect time $T_i$ for an object $i$ in the system is the random variable with the phase type probability distribution with representation $(\pi^i, A)$.*

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$$\pi^i = (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_{n-1}, 0, \ldots, 0, \pi_i),$$

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$A$ is a matrix in the similar form as the matrix $S$ in the previous Propositions: in **the One-way Model**

$$\mathbf{A} = \begin{pmatrix} \mathbf{D}_{1,1} & \mathbf{P}_{1,2} & \mathbf{O}_{1,3} & \cdots & \mathbf{O}_{1,n-1} \\ \mathbf{O}_{2,1} & \mathbf{D}_{2,2} & \mathbf{P}_{2,3} & \cdots & \mathbf{O}_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O}_{n-1,1} & \mathbf{O}_{n-1,2} & \mathbf{O}_{n-1,3} & \cdots & \mathbf{D}_{n-1,n-1} \end{pmatrix}$$

where
$P_{i,j}$ is a rectangular matrix of size $\binom{n}{i} \times \binom{n}{j}$, $O_{ij}$ is a null matrix, $D_{ii}$ are square diagonal matrices for $i = 1, \dots, n$. Moreover $D_{i,i} = -(e.P'_{i,i+1}.I_{i,i})$, where $e$ is the row vector of $\binom{n}{i}$ ones and $I_{i,i}$ is the unit matrix of size $i$.

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$A$ is a matrix in the similar form as the matrix $S$ in the previous Propositions: in **the Repairable Model**

$$\mathbf{A} = \begin{pmatrix} \mathbf{D}_{0,0} & \mathbf{P}_{0,1} & \mathbf{O}_{0,2} & \cdots & \mathbf{O}_{0,n-1} \\ \mathbf{R}_{1,0} & \mathbf{D}_{1,1} & \mathbf{P}_{1,2} & \cdots & \mathbf{O}_{1,n-1} \\ \mathbf{0}_{2,0} & \mathbf{R}_{2,1} & \mathbf{D}_{2,2} & \cdots & \mathbf{O}_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O}_{n-1,0} & \mathbf{O}_{n-1,1} & \mathbf{O}_{n-1,2} & \cdots & \mathbf{D}_{n-1,n-1} \end{pmatrix}.$$

where

$R_{j,i}$ are rectangular matrices of size $\binom{n}{j} \times \binom{n}{i}$, and

$D_{i,i} = -(e^i.R'_{i-1,i} + e^{i+1}.P'_{i,i+1}).I_{i,i}$

# Renewal period

Under the assumptions of **model 2**, the process can be understood as renewal process. The renewal occurs when the process reaches the state $\omega^0$ and the renewal period covers the time which the process needs to return to the state $\omega^0$ again, if it started in $\omega^0$.

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$\Rightarrow$ We can describe the distribution of renewal period using PH-distribution.

# Renewal period

## Proposition

*In the model 2, the distribution of renewal period can be described by PH-distribution with the representation $(\pi^0, W)$, where*

$$W = \begin{pmatrix} \mathbf{D_{0,0}} & \mathbf{P_{0,1}} & \mathbf{O_{0,2}} & \mathbf{O_{0,3}} & \cdots & \mathbf{0} \\ \mathbf{O_{1,0}} & \mathbf{D_{1,1}} & \mathbf{P_{1,2}} & \mathbf{O_{1,3}} & \cdots & \mathbf{O_{1,n}} \\ \mathbf{O_{2,0}} & \mathbf{R_{2,1}} & \mathbf{D_{2,2}} & \mathbf{P_{2,3}} & \cdots & \mathbf{O_{2,n}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{O_{n,1}} & \mathbf{O_{n,2}} & \mathbf{O_{n,3}} & \cdots & \mathbf{D_{n,n}} \end{pmatrix}$$

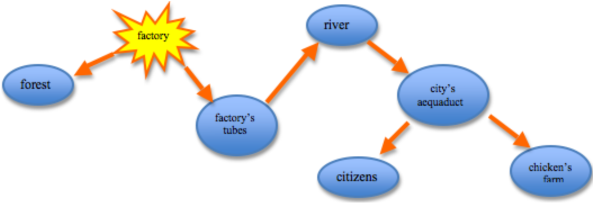
*where $P_{i,j}$ is a rectangular matrix of size $\binom{n}{i} \times \binom{n}{j}$, the symbol $O_{ij}$ denotes a null matrix and $D_{ii}$ are square diagonal matrices for $i = 1, \ldots, n$. Moreover,*

$$-D_{i,i} = e.P'_{i,i+1}.I_{i,i}$$

*where $I_{i,i}$ is the unit matrix of size $i$, $e$ is the row vector of $\binom{n}{j}$ ones.*

# Conclusions

- There has been proposed the model for deterioration spreading in two variants:



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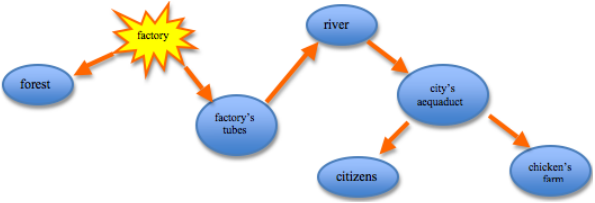

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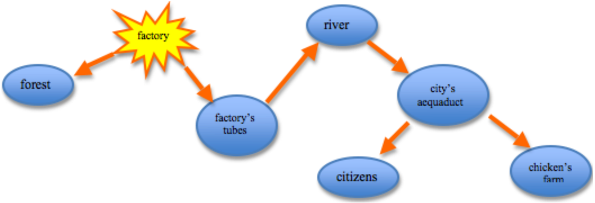

# Conclusions

First-passage time analysis for Markovian deteriorating model

Gejza Dohnal



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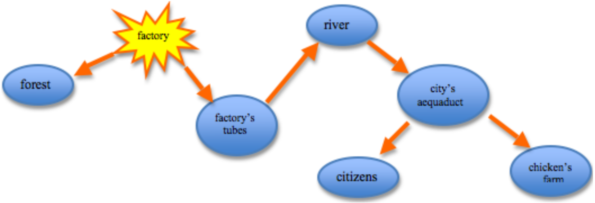

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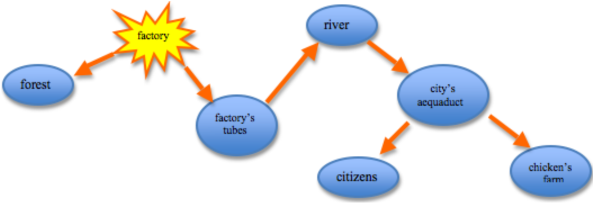

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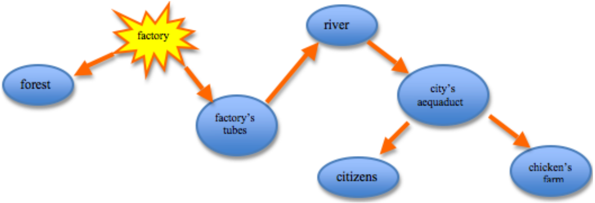


Thanks for your attention.

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